Recall from last time:

$E \to \mathbb{F}_q((\pi))$ $\quad \mathbb{F}_q = \mathcal{O}_E/\pi$

$E \to [E:\mathbb{Q}_q] < +\infty$

$F|\mathbb{F}_q$ perfectoid field

$\longmapsto$ Curve$/E$ $\quad$ adic: $X^{ad} = Y/\varphi^{\mathbb{Z}}$

$$\begin{array}{c} Y/\varphi^{\mathbb{Z}} \\ \downarrow \\ \mathrm{Spa}(E) \end{array}$$

Y = E-Stein space

* $E = \mathbb{F}_q((\pi))$ $\quad Y = \mathbb{D}^*_F = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_F$

* $E|\mathbb{Q}_p$ $\quad \mathcal{O}(Y)^+ = W_{\mathcal{O}_E}(\mathcal{O}_F)$

$$= \left\{ \sum_{n \geq 0} [x_n]\pi^n \,\middle|\, x_n \in \mathcal{O}_F \right\}$$

$\mathcal{O}(Y)$ = Frechet completion of $\mathcal{O}(Y)^+\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$

$\varpi \in F, \quad 0 < |\varpi| < 1$

$$\boxed{\mathcal{O}(1) = \text{line bundle on } X^{ad}}$$

trivial when pulled back to $Y \twoheadrightarrow X^{ad}$
automorphy factor $\varphi \mapsto \pi^{-1}$

* Schematical curve: $X = \mathrm{Proj}\left(\bigoplus_{d\geq 0} \underbrace{H^0(X^{ad}, \mathcal{O}(d))}_{\underbrace{B^{\varphi=\pi^d}}_{\infty\text{-dim. } E\text{-Banach space}}}\right)$

$B := \mathcal{O}(Y) =$ Fontaine's type period ring.

$X =$ Dedekind scheme + morphism of ringed spaces

$X^{ad} \to X$.

classical Tate points

$$\underbrace{|X^{ad}|^{cl}}_{\substack{\| \\ |Y|^{cl}/\varphi^{\mathbb{Z}}}} \xrightarrow{\sim} \underbrace{|X|}_{\text{closed points}}$$

$|Y|^{cl} = \{V(\xi) \mid \xi \in \mathcal{O}(Y)^+ \text{ primitive irreducible}\}$

$x \in |X^{ad}|^{cl}$ $\quad x \mapsto x' \in |X|$ $\quad k(x) = k(x')$ perfectoid $|E$

$\underbrace{[k(x)^\flat : F]}_{\substack{\| \\ \deg(x)}} < +\infty$

alg. closed if F alg. closed

1 if F is alg. closed

$$\widehat{\mathcal{O}_{X,k'}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X^{ad},k}} = B^+_{dR}(\mathcal{C}(k))$$

Suppose F is alg. closed:

$t \in H^0(X, \mathcal{O}(1)) \setminus \{0\}$

$\infty := V(t)$ (any point is like that)

$$X \setminus \{\infty\} = \operatorname{Spec}\Big(\underbrace{B\big[\tfrac{1}{t}\big]^{\varphi = \mathrm{Id}}}_{B_e = \text{P.I.D.}}\Big)$$

$(B_e, -\mathrm{ord}_\infty)$ not euclidean $\underbrace{\text{almost euclidean}}_{\substack{\forall x,y\ \exists a,b\ x = ay+b \\ \deg(b) \leq \deg(y)}}$

Vector bundles: * GAGA:

$X^{ad} \to X$ induces $\boxed{v.b./X \xrightarrow{\sim} v.b./X^{ad}}$

$\to$ focus on v.b./X.

* $\forall f \in E(X)^\times$, $\deg(\mathrm{div} f) = 0 \Rightarrow$ one can ~~[illegible]~~ define $\deg(\mathcal{E})$ for

any v.b. $\mathcal{E}/X$ and $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$ if $\mathcal{E} \neq 0$.

F alg. closed:

$$B_e = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X) \text{ for any } \infty \in |X|$$

B_e $\|$ P.I.D. $\qquad B^+_{dR} = \widehat{\mathcal{O}_{X,\infty}}$

$\Rightarrow$ groupoid of rk. n v.b. $/X = \left[GL_n(B_e) \backslash GL_n(B_{dR}) / GL_n(B^+_{dR}) \right]$

Line bundles:

B_e is a P.I.D. $\Leftrightarrow$ $\mathrm{Pic}(X) \xrightarrow[\deg]{\sim} \mathbb{Z}$ $\qquad \left(\mathrm{Pic}^0(X) = Cl(B_e) \right)$

$[\mathcal{O}_X(d)] \leftarrow d$

$*$ $\lambda = \frac{d}{h} \in \mathbb{Q}$, $(d,h) = 1$

alg. closed

$E_h | E$ unramified degree h residue field $F^{\mathrm{Frob}_q^h} =: \mathbb{F}_{q^h}$

$$X \otimes_E E_h = X_{E_h}$$

$$\Big\downarrow \pi_h \quad \text{cyclic covering}$$

$$X := X_E$$

③

$$\varphi_{E_h} = \varphi_E^h$$

$$\begin{array}{ccc} X^{ad}_{E_h} & = & Y_E / \varphi_E^{h\mathbb{Z}} \\ \downarrow \pi_h^{ad} & & \downarrow \\ X^{ad}_E & = & Y_E / \varphi_E^{\mathbb{Z}} \end{array}$$

partial unfolding of the Frobenius cover $= \pi_h$

Def: $\mathcal{O}_{X_E}(\lambda) := \underbrace{\pi_{h*} \mathcal{O}_{X_{E_h}}(d)}_{\text{v.b. } h \text{ degree } d}$ – stable of slope λ.

Th: (1) Any slope λ s.s. v.b. is $\simeq$ to $\mathcal{O}(\lambda)^{\oplus \text{finite}}$

(2) The H.N. filtration of a v.b. is split

(3) $$\{\lambda_1 \geq \dots \geq \lambda_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\sim} \{\text{v.b.}/X\}/\sim$$
$$(\lambda_1, \dots, \lambda_n) \longmapsto \left[\bigoplus_i \mathcal{O}(\lambda_i)\right]$$

Rem: $(1)+(2) \Leftrightarrow (3)$ but in fact $(1) \Rightarrow (2)$ easily since

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = H^1(X, \mathcal{O}(-\lambda) \otimes \mathcal{O}(\mu))$$
$$= \bigoplus_{\text{finite}} H^1(X, \mathcal{O}(\mu - \lambda)) = 0 \text{ if } \lambda \leq \mu.$$

→ difficult point = proove (1) - Uses periods of p-divisible groups $/ \mathcal{O}_C$ - $C|\mathbb{Q}_p$ alg. closed

<u>Rem</u>: v.b. $/X \xrightarrow{\sim} \{(M, W, u)\}$

free B_e-module

free B_{dR}^+-module

$u : M \otimes_{B_e} B_{dR} \xrightarrow{\sim} W[\frac{1}{t}]$

$$\mathcal{E} \longmapsto \left(\Gamma(X \setminus \infty, \mathcal{E}), \widehat{\mathcal{E}}_\infty, \mathrm{Can}\right)$$

$$R\Gamma(X, \mathcal{E}) = \big[\underset{0}{M \oplus W} \longrightarrow \underset{1}{W[\tfrac{1}{t}]}\big]$$

$$(x, y) \longmapsto u(x \otimes 1) - y$$

In particular $H^1(X, \mathcal{O}(-1)) = B_{dR} / tB_{dR}^+ + B_e$

$\neq 0$

$\Updownarrow$ $(B_e, -\mathrm{ord}_\infty)$ not euclidean

$$H^1(X, \mathcal{O}_X) = B_{dR} / B_{dR}^+ + B_e = 0$$

$\Updownarrow$

$(B_e, -\mathrm{ord}_\infty)$ almost euclidean

4

$(B_e, -\mathrm{ord}_\infty)$ not euclidean $\iff \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) \neq 0$

One easily checks that if

$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{O} \to 0$ is non split

Then $\mathcal{E}$ is stable of slope $\frac{1}{2}$.

(and in fact $\simeq \mathcal{O}(\frac{1}{2})$ according to the classification theorem (but we don't need it to check this))

X	$\mathbb{P}^1_k$
$H^1(\mathcal{O}(-1)) \neq 0$	$H^1(\mathcal{O}(1)) = 0$
$(B_e, -\mathrm{ord}_\infty)$ non-euclidean	$(k[T], \deg)$ euclidean
$\exists\, \mathcal{E}$ stable$/X$ s.t. $\mu(\mathcal{E}) \notin \mathbb{Z}$	$\forall\, \mathcal{E}$ stable $/\mathbb{P}^1$ $\mu(\mathcal{E}) \in \mathbb{Z}$.

An application: **Th:** F alg. closed. Then $X \otimes_E \bar{E}$ is simply connected i.e. $\pi_1(X) = \mathrm{Gal}(\bar{E}|E)$.

dem:

$$\begin{tikzcd} X' \arrow[d, "\text{étale finite}"] \\ X \end{tikzcd} \qquad X' = \mathrm{Spec}(\mathcal{A})$$

locally free of finite rk $\mathcal{O}_X$-algebra

X'/X étale $\Longleftrightarrow$ $\mathrm{tr}_{\mathcal{A}/\mathcal{O}_X} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{O}_X$ trace quad. form is perfect.

$\Longrightarrow$ $\mathcal{A} \simeq \check{\mathcal{A}}$ as a v.b./X.

$\Longrightarrow$ If $\lambda \in \mathbb{Q}$ shows up as a slope of $\mathcal{A}$ then $-\lambda$ too.

$\Longrightarrow$ if $\mathcal{A}$ is not ss of slope 0 then $\exists \lambda > 0$ s.t. $\mathcal{O}(\lambda) \subset \mathcal{A}$

direct factor

$$\underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n \text{ factors}} \xrightarrow{\text{mult. morphism}} \mathcal{A}$$

$$\mathcal{O}(\lambda) \otimes \cdots \otimes \mathcal{O}(\lambda) \longrightarrow 0$$ if $n >> 0$ since

$\mathrm{Hom}(\mathcal{O}(n\lambda), \mathcal{A}) = 0$ for $n >> 0$.

$\Longrightarrow$ $\Gamma(X, \mathcal{O}(\lambda)) \subset \Gamma(X, \mathcal{A})$ is made of nilpotent elements

Impossible since X' is reduced.

$\Longrightarrow$ $\mathcal{A}$ is s.s. slope 0.

$$\mathrm{Vect}_E \xrightarrow[\sim]{\otimes\text{-equivalence}} \text{Slope } 0 \text{ s.s. v.b.}/X$$

$$V \longmapsto V \underset{E}{\otimes} \mathcal{O}_X$$

$$H^0(X, \mathcal{E}) \longleftarrow\!\!\shortmid \mathcal{E}$$

$$\Rightarrow \mathcal{A} = \underbrace{H^0(X, \mathcal{A})}_{\text{finite étale } E\text{-alg.}} \underset{E}{\otimes} \mathcal{O}_X$$

$$\Rightarrow X' = X \underset{\mathrm{Spec}\, E}{\times} T \qquad T/\mathrm{Spec}(E) \text{ finite étale} \quad \square$$

The case when F is not alg. closed

$$X_{\hat{\bar{F}}} \circlearrowleft \mathrm{Gal}(\bar{F}|F) =: \Gamma$$

$$\downarrow \alpha$$

$$X_F$$

$x \in |X_{\hat{\bar{F}}}|$ = closed points

$$\alpha(x) = \begin{cases} \text{generic point if } |\Gamma . x| = \infty \\ \text{closed point if } |\Gamma . x| < +\infty \end{cases}$$

$y \in |X_F| \qquad \alpha^{-1}(y) = \Gamma\text{-orbit.}$

$$|X_F| = |X_{\hat{F}}|^{\text{finite } \Gamma\text{-orbit}} / \Gamma .$$

Th [Narasimhan - Seshadri type result]

Continuous p-adic rep.

$$\text{Slope } 0 \text{ s.s. v.b.} / X_F \xrightarrow{\sim} \mathrm{Rep}_E(\mathrm{Gal}(\overline{F}|F))$$

$$\mathcal{E} \longmapsto H^0(X_{\hat{\overline{F}}}, \alpha^* \mathcal{E})$$

$$\Rightarrow \pi_1(X_F) = \mathrm{Gal}(\overline{F}|F) \times \mathrm{Gal}(\overline{E}|E)$$

(same type of proof as before).